Concentration inequalities for disordered models

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Abstract

We use a generalization of Hoeffding's inequality to show concentration results for the free energy of disordered pinning models, assuming only that the disorder has a finite exponential moment. We also prove some concentration inequalities for directed polymers in random environment, which we use to establish a large deviations result for the end position of the polymer under the polymer measure.

Key words. Concentration, martingale differences, disordered pinning model, directed polymers in random environment, large deviations

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1 Introduction

Let $S = (S_n)_{n\in\mathbb{N}}$ be a symmetric random walk defined on a probability space $(\Sigma, \mathcal{E}, \mathbf{P})$ with values in $\mathbb{Z}^d$, starting at 0. Let I be a denumerable set, and let $\eta = (\eta_k)_{k\in I}$ be a sequence of i.i.d. random variables defined on another probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expectation of a function f with respect to the probability measure $\mathbf{P}$ (respectively $\mathbb{P}$) will be denoted by $\mathbf{E}f = \int_\Sigma f d\mathbf{P}$ (respectively $\mathbb{E}f = \int_\Omega f d\mathbb{P}$). We assume that there exists $\beta > 0$ such that $\mathbb{E}e^{\beta|\eta_i|} < \infty$ for $i \in I$. For every fixed realization of η, we consider functionals of the form

$$Y_n^\eta = \mathbf{E}\exp(H_n^\eta(S))f_n(S_n), \tag{1.1}$$

where H_n^η is a function of the first n terms of S, and f_n is a bounded positive function defined on $\mathbb{R}^d$ with $\mathbf{P}(f_n(S_n) > 0) \neq 0$. When

$$I = \mathbb{N},\ H_n^\eta(S) = \sum_{k=1}^n (\beta\eta_k + h)\mathbf{1}_{\{S_k=0\}},\ \text{and } f_n(x) = \mathbf{1}_{\{x=0\}},$$

Y_n^η is the partition function of the disordered pinning model studied in [5], [6]. When

$$I = \mathbb{N}\times\mathbb{Z}^d,\ H_n^\eta(S) = \beta\sum_{j=1}^n \eta_{(j,S_j)},\ \text{and } f_n = 1,$$

Y_n^η is the partition function of a directed polymer in a random environment. In this paper we use the exponential inequality for martingales that we obtained in [10] to prove some concentration properties of the free energy for both models.

Let us present the two models in more detail. We refer to [5], [6], for a survey on disordered pinning models. Let $S = (S_n)_{n\in\mathbb{N}}$ be a random walk on $\mathbb{Z}^d$ starting at 0, defined on a probability space $(\Sigma, \mathcal{E}, \mathbf{P})$. We suppose that the increment variables $(S_n - S_{n-1})_{n\geq 1}$ are independent, symmetric, and have finite variance. For each $\beta \geq 0$, $h \in \mathbb{R}$ and $\eta = (\eta_k)_{k\in\mathbb{N}}$ a sequence of real numbers, we define the Gibbs measure $\mathbf{P}_n$ on Σ by giving its density:

$$\frac{d\mathbf{P}_n}{d\mathbf{P}} = \frac{1}{Z_n}\exp(\sum_{k=1}^{n}(\beta\eta_k + h)\mathbf{1}_{\{S_k=0\}})\mathbf{1}_{\{S_n=0\}}, \tag{1.2}$$

where Z_n is the partition function

$$Z_n = \mathbf{E}\exp(\sum_{k=1}^{n}(\beta\eta_k + h)\mathbf{1}_{\{S_k=0\}})\mathbf{1}_{\{S_n=0\}}. \tag{1.3}$$

The set $\tau = \{n \geq 0 : S_n = 0\}$ of visits to 0 is then a renewal process. It means that if $\tau = \{\tau_0, \tau_1, \cdots\}$ with $\tau_0 = 0$ and $\tau_i > \tau_{i-1}$, then $(\tau_i - \tau_{i-1})_{i\geq 1}$ is an i.i.d. sequence of positive random variables. Let $\delta_n = \mathbf{1}_{\{n\in\tau\}}$. Then we can write

$$\frac{d\mathbf{P}_n}{d\mathbf{P}} = \frac{1}{Z_n}\exp(\sum_{k=1}^{n}(\beta\eta_k + h)\delta_k)\delta_n, \tag{1.4}$$

and

$$Z_n = \mathbf{E}\exp(\sum_{k=1}^{n}(\beta\eta_k + h)\delta_k)\delta_n, \tag{1.5}$$

without mentioning trajectories S anymore. In the following we consider a renewal process $\tau = (\tau_j)_{j\in\mathbb{N}}$ on a probability space $(\Sigma, \mathcal{E}, \mathbf{P})$ with values in $\mathbb{N}$, starting with $\tau_0 = 0$, and a realization η of an i.i.d. sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity we write $\mathbb{E}f(\eta) = \mathbb{E}f(\eta_0)$ for any f such that $f \circ \eta_0$ is integrable. We fix $\beta > 0$ and assume that $\mathbb{E}e^{\beta|\eta|} < \infty$. We denote by $\lambda(\beta) = \ln \mathbb{E}e^{\beta\eta}$ the logarithmic moment generating function of η. Thanks to subadditivity properties it is easily shown that $\frac{1}{n}\mathbb{E}\ln Z_n$ converges to a number $F(\beta, h)$ which is called the free energy of the model, and it is also known that $\frac{1}{n}\ln Z_n$ converges $\mathbb{P}$ a.s. towards $F(\beta, h)$. The function $\eta \mapsto \ln Z_n$ is Lipschitz with constant $C = \beta\sqrt{n}$, so if the disorder η is gaussian, then one can use the concentration property of gaussian measures to deduce that there exist constants c_1, c_2 such that

$$\forall x > 0,\ \mathbb{P}(\frac{1}{n}|\ln Z_n - \mathbb{E}\ln Z_n| > x) \leq c_1\exp(-c_2\frac{nx^2}{\beta^2}). \tag{1.6}$$

This inequality remains valid if η_0 is a bounded variable or if the law of η_0 satisfies a log-Sobolev inequality (see [5] for more details). We refer to [8] for a survey of the concentration measure phenomenon. Here we show that there is no need to make such assumptions to get exponential concentration of the free energy. Assuming only that $\mathbb{E}e^{\beta|\eta|} < +\infty$, we prove that the inequality (1.6) is always valid for small values of x.

THEOREM 1.1 *Assume that $\mathbb{E}e^{\beta|\eta|} < +\infty$ and set $K = 2\max(e^{h+\lambda(\beta)}, 1) \times \max(e^{-h+\lambda(-\beta)}, 1)$. Then for all $n \geq 1$,*

$$\mathbb{E}\exp(\pm t(\ln Z_n - \mathbb{E}\ln Z_n)) \leq \exp(nKt^2) \text{ for all } t \in [0,1] \tag{1.7}$$

and

$$\mathbb{P}(\pm\frac{1}{n}(\ln Z_n - \mathbb{E}\ln Z_n) > x) \leq \begin{cases} \exp(-\frac{nx^2}{4K}) & \textit{if } x \in (0, 2K], \\ \exp(-n(x-K)) & \textit{if } x \in (2K, \infty). \end{cases} \tag{1.8}$$

We refer to [4] for a review of directed polymers in random environment. In this model we consider $S = (S_n)_{n\in\mathbb{N}}$ the simple random walk on $\mathbb{Z}^d$ starting at 0, defined on a probability space $(\Sigma, \mathcal{E}, \mathbf{P})$. Let $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ be a sequence of real-valued, non-constant and i.i.d. random variables defined on another probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The path S represents the directed polymer and η the random environment. For $n > 0$, let Σ_n be the set of paths of length n, and define the random polymer measure $\mathbf{P}_n$ on the path space $(\Sigma, \mathcal{E})$ by

$$\frac{d\mathbf{P}_n}{d\mathbf{P}}(S) = \frac{1}{Z_n}\exp(\beta H_n(S)), \tag{1.9}$$

where $\beta \in \mathbb{R}$ is the inverse temperature,

$$H_n(S) = \sum_{j=1}^{n} \eta(j, S_j), \text{ and } Z_n = \mathbf{E}\exp(\beta H_n(S)). \tag{1.10}$$

In other words, for a given realisation of the environment η, the measure $\mathbf{P}_n$ gives to a polymer chain S having an energy $-H_n(S)$ at temperature $T = \frac{1}{\beta}$, a weight proportional to $e^{\beta H_n(S)}$ (configurations of lowest energy are the most probable). For simplicity we write $\mathbb{E}f(\eta) = \mathbb{E}f(\eta(0,0))$ for any f such that $f \circ \eta(0,0)$ is integrable. Let $\lambda(\beta) = \ln \mathbb{E}e^{\beta\eta}$ be the logarithmic moment generating function of η. We shall use the point to point partition function:

$$Z_n(x,y) = Z_n(x,y;\eta) = \mathbf{E}^x \exp(\beta\sum_{j=1}^{n}\eta(j,S_j))\mathbf{1}_{S_n=y}, \tag{1.11}$$

where $\mathbf{E}^x$ is the expectation with respect to $\mathbf{P}^x$, the measure of the random walk starting from x. The Markov Property of the simple random walk yields that

$$Z_{n+m}(x,z) = \sum_y Z_n(x,y;\eta)Z_m(y,z;\Theta_n\eta), \tag{1.12}$$

where Θ_n denotes the shift operator

$$\Theta_n\eta(i,x) = \eta(i+n,x).$$

It is well known that this implies that the sequence $\mathbb{E}\ln Z_n$ is superadditive, hence the limit

$$p(\beta) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\ln(Z_n) = \sup_n \frac{1}{n}\mathbb{E}\ln(Z_n) \in (-\infty, \lambda(\beta)] \tag{1.13}$$

exists. It is called the free energy of the polymer. The function $\eta \mapsto \ln Z_n$ is Lipschitz with constant $C = \beta\sqrt{n}$, so if the disorder η is gaussian, then the concentration inequality (1.6) holds true ([2], Proposition 2.3), and therefore $\frac{1}{n}\ln Z_n$ converges $\mathbb{P}$ a.s. towards $p(\beta)$. Using an inequality due to Lesigne and

Volny ([9]), Comets, Shiga and Yoshida ([3]) proved that if $\mathbb{E}e^{\beta|\eta|} < +\infty$ for all $\beta > 0$, then for every $x > 0$, there exists $n_0 \in \mathbb{N}^*$ such that for any $n \geq n_0$,

$$\mathbb{P}(|\frac{1}{n}\ln Z_n - \frac{1}{n}\mathbb{E}\ln Z_n| > x) \leq \exp(-\frac{n^{\frac{1}{3}}x^{\frac{2}{3}}}{4}), \tag{1.14}$$

which again allows them to prove that $\frac{1}{n}\ln Z_n$ converges $\mathbb{P}$ a.s. towards $p(\beta)$. We improved this result in [10] by showing that if $\mathbb{E}e^{\beta|\eta|} < \infty$ for a fixed $\beta > 0$, then there exists $K > 0$ such that for all $n \geq 1$,

$$\mathbb{P}(\pm\frac{1}{n}(\ln Z_n - \mathbb{E}\ln Z_n) > x) \leq \begin{cases} \exp(-\frac{nx^2}{4K}) & \text{if } x \in (0, 2K], \\ \exp(-n(x-K)) & \text{if } x \in (2K, \infty). \end{cases}$$

Here we extend this concentration property to functions of the form $\ln Y_n = \ln \mathbf{E} f_n(S_n)\exp(\beta H_n(S))$, where (f_n) is any sequence of bounded positive functions such that $\mathbf{P}(f_n(S_n) > 0) \neq 0$.

Theorem 1.2 *Assume that* $\mathbb{E}e^{\beta|\eta|} < +\infty$ *and set* $K = 2\exp(\lambda(-\beta)) + \lambda(\beta))$. *Let* f_n *be a sequence of bounded positive functions on* $\mathbb{R}^d$ *such that for all* $n \geq 1$, $\mathbf{P}(f_n(S_n) > 0) \neq 0$. *Set* $Y_n = \mathbf{E}f_n(S_n)e^{\beta H_n(S)}$. *Then for all* $n \geq 1$,

$$\mathbb{E}e^{\pm t(\ln Y_n - \mathbb{E}\ln Y_n)} \leq \exp(nKt^2) \text{ for all } t \in [0,1], \tag{1.15}$$

and

$$\mathbb{P}(\pm\frac{1}{n}(\ln Y_n - \mathbb{E}\ln Y_n) > x) \leq \begin{cases} \exp(-\frac{nx^2}{4K}) & \text{if } x \in (0, 2K], \\ \exp(-n(x-K)) & \text{if } x \in (2K, \infty). \end{cases} \tag{1.16}$$

In [2], Carmona and Hu consider a gaussian environment and show among other things a large deviations result for the end position S_n of the polymer under the polymer measure $\mathbf{P}_n$ ([2], Theorems 1.1 and 1.2). Their proof relies on the concentration inequality (1.6). Here we prove that their large deviations result holds for every environment such that $\mathbb{E}e^{\beta|\eta|} < \infty$. Let $B_d = \{x \in \mathbb{R}^d; \|x\|_1 = |x_1| + \cdots + |x_d| \leq 1\}$ be the closed unit ball of $\mathbb{R}^d$ in the l^1-norm, $\mathring{B}_d$ be the corresponding open ball.

Theorem 1.3 *Let* $\beta > 0$ *and assume that* $\mathbb{E}e^{\beta|\eta|} < +\infty$. *Then there exists a convex rate function* $I_\beta : B_d \to [0, \ln(2d) + p(\beta)]$ *such that* $\mathbb{P}$ *a.e.,*

$$\limsup_{n\to\infty} \frac{1}{n}\ln \mathbf{P}_n(\frac{S_n}{n} \in F) \leq -\inf_{x\in F} I_\beta(x) \text{ for } F \text{ closed } \subset B_d, \tag{1.17}$$

$$\liminf_{n\to\infty} \frac{1}{n}\ln \mathbf{P}_n(\frac{S_n}{n} \in G) \geq -\inf_{x\in G} I_\beta(x) \text{ for } G \text{ open } \subset B_d. \tag{1.18}$$

Theorem 1.4 *Let* $\beta > 0$ *and assume that* $\mathbb{E}e^{\beta|\eta|} < +\infty$. *Then for every* $x \in \mathring{B}_d \cap \mathbb{Q}^d$,

$$\lim_{n\to\infty} \frac{1}{n}\ln \mathbf{P}_n(\frac{S_n}{n} = x) = -I_\beta(x) \ \mathbb{P} \text{ a.s.}, \tag{1.19}$$

where we take the limit along n *such that* $\mathbf{P}(S_n = nx) > 0$. *Moreover,* $I_\beta(0) = 0$, $I_\beta(x_1, \cdots, x_d) = I_\beta(\pm x_{\sigma(1)}, \cdots, \pm x_{\sigma(d)})$ *for any permutation* σ *of* $\{1, \cdots, d\}$, *and for* $e_1 = (1, 0, \cdots, 0) \in \mathbb{Z}^d$, *we have*

$$I_\beta(e_1) = \ln(2d) + p(\beta). \tag{1.20}$$

The paper is organized as follows. In a first part we recall in a simpler form a theorem of [10], which gives optimal conditions on the martingale M_n to provide exponential bounds for $\mathbb{P}(\frac{M_n}{n} > x)$. In the second part we prove Theorem 1.1 by writing $\ln Z_n - \mathbb{E}\ln Z_n$ as a sum of $(\mathcal{F}_j)_{1\le j\le n}$ martingale differences, where $\mathcal{F}_j = \sigma(\eta_i : 1 \le i \le j)$, and showing that the conditional exponential moments of the martingale differences are uniformly bounded. In the third part we prove in a similar manner Theorem 1.2 by writing $\ln Y_n - \mathbb{E}\ln Y_n$ as a sum of $(\mathcal{F}_j)_{1\le j\le n}$ martingale differences, using this time the filtration $\mathcal{F}_j = \sigma(\eta(i,x) : 1 \le i \le j, x \in \mathbb{Z}^d)$. In the last part we explain how to use our concentration results (1.16) of Theorem 1.2 to prove Theorem 1.3 and Theorem 1.4.

2 Exponential inequalities for martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{F}_0 = \{\emptyset, \Omega\} \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$ be an increasing sequence of sub-σ-fields of $\mathcal{F}$. Let $X_1, ..., X_n$ be a sequence of real-valued martingale differences defined on $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to the filtration $(\mathcal{F}_k)$: that is, for each $1 \le k \le n$, X_k is $\mathcal{F}_k$ measurable and $\mathbb{E}(X_k|\mathcal{F}_{k-1}) = 0$. Set

$$M_n = X_1 + ... + X_n. \tag{2.1}$$

If the martingale differences X_i are uniformly bounded by a constant a, then Azuma-Hoeffding's inequality ([7],[1]) states that

$$\mathbb{P}(\frac{M_n}{n} > x) \le e^{-\frac{nx^2}{a^2}}. \tag{2.2}$$

Lesigne and Volný proved ([9]) that if for some constant $K > 0$ and all $k = 1, ..., n$,

$$\mathbb{E}e^{|X_k|} \le K, \tag{2.3}$$

then for any $x > 0$,

$$\mathbb{P}(\frac{M_n}{n} > x) = O(e^{-\frac{1}{4}x^{2/3}n^{1/3}}). \tag{2.4}$$

They also showed that this is the best possible inequality that we can have under the condition (2.3), even in the class of stationary and ergodic sequences of martingale differences, in the sense that there exist such sequences of martingale differences (X_i) satisfying (2.3) for some $K > 0$, but

$$\mathbb{P}(\frac{M_n}{n} > 1) > e^{-cn^{1/3}} \tag{2.5}$$

for some constant $c > 0$ and infinitely many n. We showed in [10] that the best condition for having an exponential inequality of the form

$$\mathbb{P}(\frac{M_n}{n} > x) \le Ce^{-c(x)n}$$

is to replace the expectation in (2.3) by the conditional one given $\mathcal{F}_{k-1}$. We recall rapidly one of our results, presented in a slightly simpler form, and its proof ([10], Theorem 2.1).

Theorem 2.1 *Let $(X_i)_{1\le i\le n}$ be a finite sequence of martingale differences. If for some constant $K>0$ and all $i=1,...,n$,*

$$\mathbb{E}(e^{|X_i|}|\mathcal{F}_{i-1}) \le K \quad a.s., \tag{2.6}$$

then:

$$\mathbb{E}e^{\pm tM_n} \le \exp(nKt^2) \text{ for all } t\in[0,1], \tag{2.7}$$

and

$$\mathbb{P}(\frac{\pm M_n}{n} > x) \le \begin{cases} \exp(-\dfrac{nx^2}{4K}) & \text{if } x\in(0,2K], \\ \exp(-n(x-K)) & \text{if } x\in(2K,\infty). \end{cases} \tag{2.8}$$

We need the following simple lemma.

Lemma 2.2 *Let X be a real-valued random variable defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, with $\mathbb{E}X\le 0$ and $\mathbb{E}e^{|X|}\le K$ for some $K>0$. Then for all $t\in[0,1]$,*

$$\mathbb{E}e^{tX} \le \exp(Kt^2). \tag{2.9}$$

Proof. Let $t\in[0,1]$. Since $\mathbb{E}X\le 0$, we have:

$$\begin{aligned}\mathbb{E}e^{tX} &= \sum_{k=0}^{\infty} t^k\mathbb{E}\frac{X^k}{k!} \le 1+\sum_{k=2}^{\infty} t^k\mathbb{E}\frac{X^k}{k!} \\ &\le 1+t^2\sum_{k=2}^{\infty}\mathbb{E}\frac{|X|^k}{k!} \le 1+t^2\mathbb{E}e^{|X|} \le 1+Kt^2 \le \exp(Kt^2).\end{aligned}$$

■

Proof of Theorem 2.1. By Lemma 2.2, we obtain that for every i and for every $t\in(0,1)$,

$$\mathbb{E}(e^{tX_i}|\mathcal{F}_{i-1}) \le \exp(Kt^2) \ a.s.$$

By induction on n we get immediately (2.7) for $+M_n$. Then $\forall x>0$, $\forall t\in[0,1]$,

$$\mathbb{P}(\frac{M_n}{n} > x) = \mathbb{P}(e^{tM_n} > e^{tnx}) \le e^{-ntx}\mathbb{E}e^{tM_n} \le \exp(-n(tx-Kt^2)).$$

As

$$\sup_{t\in[0,1]}(tx-Kt^2) = \begin{cases} \dfrac{x^2}{4K} & \text{if } x\in(0,2K], \\ x-K & \text{if } x\in(2K,\infty), \end{cases} \tag{2.10}$$

we obtain (2.8) for $+M_n$. We apply the same argument to the sequence $(-X_i)$ and obtain the full inequalities (2.7) and (2.8). ■

3 Disordered pinning model: proof of Theorem 1.1

We write $\ln Z_n - \mathbb{E}\ln Z_n$ as a sum of $(\mathcal{F}_j)_{1\le j\le n}$ martingale differences:

$$\ln Z_n - \mathbb{E}\ln Z_n = \sum_{j=1}^{n} V_{n,j}, \quad \text{with } V_{n,j} = \mathbb{E}_j\ln Z_n - \mathbb{E}_{j-1}\ln Z_n, \tag{3.1}$$

where $\mathbb{E}_j$ denotes the conditional expectation with respect to $\mathbb{P}$ given $\mathcal{F}_j$, $\mathcal{F}_j = \sigma(\eta_i : 1 \le i \le j)$. Theorem 1.1 is then a direct consequence of Theorem 2.1. All we have to do is to check condition (2.6), which is done in the following lemma.

Lemma 3.1 *For every $1 \le j \le n$, we have:*

$$\mathbb{E}_{j-1} \exp(|V_{n,j}|) \le K := 2 \max(e^{h+\lambda(\beta)}, 1) \times \max(e^{-h+\lambda(-\beta)}, 1). \tag{3.2}$$

Proof. Let us define

$$Z_{n,j} = \mathbf{E} \exp(\sum_{k=1,k\neq j}^{n} (\beta\eta_k + h)\delta_k)\delta_n. \tag{3.3}$$

Since $\mathbb{E}_{j-1} \ln Z_{n,j} = \mathbb{E}_j \ln Z_{n,j}$, we have:

$$V_{n,j} = \mathbb{E}_j \ln \frac{Z_n}{Z_{n,j}} - \mathbb{E}_{j-1} \ln \frac{Z_n}{Z_{n,j}}, \tag{3.4}$$

therefore

$$\mathbb{E}_{j-1} \exp(V_{n,j}) = \exp(-\mathbb{E}_{j-1} \ln \frac{Z_n}{Z_{n,j}})\mathbb{E}_{j-1} \exp(\mathbb{E}_j \ln \frac{Z_n}{Z_{n,j}}). \tag{3.5}$$

Using Jensen's inequality and the fact that $\mathcal{F}_{j-1} \subset \mathcal{F}_j$, we get:

$$\mathbb{E}_{j-1} \exp(V_{n,j}) \le \mathbb{E}_{j-1}(\frac{Z_n}{Z_{n,j}})^{-1}\mathbb{E}_{j-1}\frac{Z_n}{Z_{n,j}}. \tag{3.6}$$

Now let us write

$$\frac{Z_n}{Z_{n,j}} = \alpha_0 + \alpha_1 \exp(\beta\eta_j + h), \tag{3.7}$$

with

$$\alpha_l = \frac{\mathbf{E} \exp(\sum_{k=1,k\neq j}^{n} (\beta\eta_k + h)\delta_k)\delta_n \mathbf{1}_{\delta_j = l}}{Z_{n,j}} \quad \text{for } l = 0, 1. \tag{3.8}$$

We consider the σ-algebra $\mathcal{F}_{n,j} = \sigma(\eta_k; 1 \le k \le n, k \neq j)$. Then $\mathcal{F}_{j-1} \subset \mathcal{F}_{n,j}$ and, therefore,

$$\begin{aligned}
\mathbb{E}_{j-1}\frac{Z_n}{Z_{n,j}} &= \mathbb{E}_{j-1}\mathbb{E}(\alpha_0 + \alpha_1 \exp(\beta\eta_j + h) \mid \mathcal{F}_{n,j}) \\
&= \mathbb{E}_{j-1}(\alpha_0 + \alpha_1 \exp(\lambda(\beta) + h)) \\
&= \mathbb{E}_{j-1}\alpha_0 + \mathbb{E}_{j-1}(\alpha_1) \exp(\lambda(\beta) + h).
\end{aligned}$$

As $\alpha_0 + \alpha_1 = 1$ we deduce that

$$\mathbb{E}_{j-1}\frac{Z_n}{Z_{n,j}} \le \max(e^{h+\lambda(\beta)}, 1). \tag{3.9}$$

For the same reason, and using the convexity of the inverse function, we obtain

$$\begin{aligned}
(\frac{Z_n}{Z_{n,j}})^{-1} &= (\alpha_0 + \alpha_1 \exp(\beta\eta_j + h))^{-1} \\
&\le \alpha_0 + \alpha_1 \exp(-\beta\eta_j - h).
\end{aligned}$$

The same reasoning as before then leads to

$$\mathbb{E}_{j-1}(\frac{Z_n}{Z_{n,j}})^{-1} \leq \max(e^{-h+\lambda(-\beta)}, 1). \tag{3.10}$$

Combining (3.6), (3.9) and (3.10) we get the inequality

$$\mathbb{E}_{j-1} \exp(V_{n,j}) \leq \max(e^{h+\lambda(\beta)}, 1) \max(e^{-h+\lambda(-\beta)}, 1). \tag{3.11}$$

The same inequality holds with $-V_{n,j}$ instead of $V_{n,j}$, from which we deduce (3.2). ■

We conclude this section by noticing that the concentration inequality (1.8) implies immediately the following convergence result.

COROLLARY 3.2 *Assume that* $\mathbb{E}e^{\beta|\eta|} < +\infty$. *Then:*

$$\frac{\ln Z_n}{n} - \mathbb{E}\frac{\ln Z_n}{n} \to 0 \quad a.s. \text{ and in } L^p, p \geq 1. \tag{3.12}$$

4 Directed polymers in random environment: proof of Theorem 1.2

The proof of Theorem 1.2 follows the same line as the proof of Theorem 1.1. We write $\ln Y_n - \mathbb{E}\ln Y_n$ as a sum of $(\mathcal{F}_j)_{1\leq j\leq n}$ martingale differences:

$$\ln Y_n - \mathbb{E}\ln Y_n = \sum_{j=1}^{n} V_{n,j}, \quad \text{with } V_{n,j} = \mathbb{E}_j \ln Y_n - \mathbb{E}_{j-1}\ln Y_n, \tag{4.1}$$

where this time $\mathbb{E}_j$ denotes the conditional expectation with respect to $\mathbb{P}$ given $\mathcal{F}_j$, $\mathcal{F}_j = \sigma(\eta(i,x) : 1 \leq i \leq j, x \in \mathbb{Z}^d)$. According to Theorem 2.1, to prove Theorem 1.2 we only have to bound the conditional exponential moments of the martingale differences, which is done in the following lemma.

LEMMA 4.1 *For every* $1 \leq j \leq n$, *we have:*

$$\mathbb{E}_{j-1}\exp(|V_{n,j}|) \leq K := 2\exp(\lambda(\beta) + \lambda(-\beta)). \tag{4.2}$$

Proof. For every $1 \leq j \leq n$, we define

$$H_{n,j}(S) = \sum_{1\leq k\leq n, k\neq j} \eta(k, S_k) \text{ and } Y_{n,j} = \mathbf{E} f_n(S_n)\exp(\beta H_{n,j}(S)). \tag{4.3}$$

Since $\mathbb{E}_{j-1}\ln Y_{n,j} = \mathbb{E}_j \ln Y_{n,j}$, we obtain as in Lemma 3.1 that

$$\mathbb{E}_{j-1}\exp(V_{n,j}) \leq \mathbb{E}_{j-1}(\frac{Y_n}{Y_{n,j}})^{-1}\mathbb{E}_{j-1}\frac{Y_n}{Y_{n,j}}. \tag{4.4}$$

Now let us write

$$\frac{Y_n}{Y_{n,j}} = \sum_{x\in\mathbb{Z}^d} \alpha_x \exp(\beta\eta(j,x)), \tag{4.5}$$

with

$$\alpha_x = \frac{\mathbf{E} f_n(S_n) e^{\beta H_{n,j}(S)} \mathbf{1}_{S_j = x}}{Y_{n,j}}. \tag{4.6}$$

We consider the σ-algebra $\mathcal{F}_{n,j} = \sigma(\eta(k,x); 1 \le k \le n, k \ne j, x \in \mathbb{Z}^d)$. Using that $\mathcal{F}_{j-1} \subset \mathcal{F}_{n,j}$, the α_x are $\mathcal{E}_{n,j}$-measurable, and the $\exp(\beta\eta(j,x))$ are independent of $\mathcal{E}_{n,j}$, we obtain:

$$\begin{aligned}
\mathbb{E}_{j-1} \frac{Y_n}{Y_{n,j}} &= \mathbb{E}_{j-1} \mathbb{E}\Big(\sum_{x \in \mathbb{Z}^d} \alpha_x \exp(\beta\eta(j,x)) \mid \mathcal{F}_{n,j}\Big) \\
&= \mathbb{E}_{j-1} \sum_{x \in \mathbb{Z}^d} \alpha_x \exp(\lambda(\beta)).
\end{aligned}$$

As $\sum_{x \in \mathbb{Z}^d} \alpha_x = 1$, we get

$$\mathbb{E}_{j-1} \frac{Y_n}{Y_{n,j}} = \exp(\lambda(\beta)). \tag{4.7}$$

The inverse function is convex and $\sum_{x \in \mathbb{Z}^d} \alpha_x = 1$, therefore

$$\begin{aligned}
\Big(\frac{Y_n}{Y_{n,j}}\Big)^{-1} &= \Big(\sum_{x \in \mathbb{Z}^d} \alpha_x \exp(\beta\eta(j,x))\Big)^{-1} \\
&\le \sum_{x \in \mathbb{Z}^d} \alpha_x \exp(-\beta\eta(j,x)).
\end{aligned}$$

The same reasoning as before then leads to

$$\mathbb{E}_{j-1} \Big(\frac{Y_n}{Y_{n,j}}\Big)^{-1} \le \exp(\lambda(-\beta)). \tag{4.8}$$

Combining (4.4), (4.7) and (4.8) we get the inequality

$$\mathbb{E}_{j-1} \exp(V_{n,j}) \le \exp(\lambda(\beta) + \lambda(-\beta)). \tag{4.9}$$

The same inequality holds with $-V_{n,j}$ instead of $V_{n,j}$, from which we deduce (4.2). ■

As in the preceding section, we notice that the concentration inequality (1.16) implies immediately that $\frac{\ln Y_n}{n} - \mathbb{E}\frac{\ln Y_n}{n}$ converges to 0 a.s. and also in L^p for all $p \ge 1$. In particular, when $f_n = 1$, we conclude that $\frac{\ln Z_n}{n}$ converges $\mathbb{P}$ a.s. towards $p(\beta)$. In the same manner, if the sequence $\mathbb{E} \ln Y_n$ is superadditve, then it converges towards a limit $L(\beta)$, and the concentration inequality (1.16) implies that $\frac{\ln Y_n}{n}$ converges $\mathbb{P}$ a.s. towards $L(\beta)$. Let us denote by $\mathbf{E}_n f = \int f d\mathbf{P}_n$ the expectation of a function f with respect to the polymer measure $\mathbf{P}_n$. In particular, with the notations of Theorem 1.2, we have $\mathbf{E}_n f_n(S_n) = \frac{Y_n}{Z_n}$, and we can deduce from Theorem 1.2 the following result, which will be used repeatedly in the next section.

Corollary 4.2 *Assume that $\mathbb{E}e^{\beta|\eta|} < +\infty$. Let (f_n) be a sequence of bounded positive functions on $\mathbb{R}^d$ such that for all $n \ge 1$, $\mathbf{P}(f_n(S_n) > 0) \ne 0$. Set $Y_n = \mathbf{E} f_n(S_n) e^{\beta H_n(S)}$. Then:*

$$\frac{\ln Y_n}{n} - \mathbb{E}\frac{\ln Y_n}{n} \to 0 \quad \textit{a.s. and in } L^p \textit{ for all } p \ge 1. \tag{4.10}$$

If moreover the sequence $\mathbb{E}\ln Y_n$ is superadditive, then the limit

$$L(\beta) = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\ln Y_n = \sup_{n\geq 1}\frac{1}{n}\mathbb{E}\ln Y_n \tag{4.11}$$

exists, so does the limit

$$\lim_{n\to+\infty}\frac{1}{n}\mathbb{E}\ln \mathbf{E}_n f_n(S_n) = L(\beta) - p(\beta), \tag{4.12}$$

and then:

$$\lim_{n\to+\infty}\frac{1}{n}\ln \mathbf{E}_n f_n(S_n) = L(\beta) - p(\beta) \quad \textit{a.s. and in } L^p, \textit{ for all } p\geq 1. \tag{4.13}$$

5 Large deviations for directed polymers in random environment

Theorems 1.3 and 1.4 are proved in [2] (Theorems 1.1 and 1.2) in the case where $(\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ are i.i.d. $\mathcal{N}(0,1)$ gaussian random variables. Their proofs rely essentially on subadditivity, and on the concentration property of the gaussian measure. We can extend it to the case of a general environment, assuming only that $\mathbb{E}e^{\beta|\eta|} < +\infty$, because the subaddivity arguments are still valid, and Theorem 1.2 provides the concentration properties we need. More precisely we shall use several times (that is for several different sequences (f_n)'s) Corollary 4.2. Without loss of generality we shall assume that $\mathbb{E}\eta = 0$. Our proof of Theorems 1.3 and 1.4 are essentially the same as Theorems 1.1 and 1.2 of [2], so we won't detail them. Instead let us give two lemmas which illustrate how Theorem 1.2 is used. In Lemma 3.1 Carmona and Hu establish that for any $\lambda > 0$, for any $x\in B_d$ and any sequence $x_n\in\mathbb{R}^d$ satisfying $x_n/n\to x$, the following limit exists thanks to subadditivity:

$$\lim_{n\to+\infty}\frac{\mathbb{E}\ln \mathbf{E}e^{-\lambda\|S_n - x_n\|_1}e^{\beta H_n(S)}}{n} = \phi_\lambda(x). \tag{5.1}$$

We first consider $f_n(z) = e^{-\lambda\|z-x_n\|_1}$, where $\lambda > 0$ and $x_n\in\mathbb{R}^d$, and applying Corollary 4.2 we obtain:

Lemma 5.1 *For any $x\in B_d$ and any sequence $x_n\in\mathbb{R}^d$ satisfying $x_n/n\to x$, the following limits exist a.s. and in L^p, for all $p\geq 1$:*

$$\lim_{n\to+\infty} -\frac{1}{n}\ln \mathbf{E}_n e^{-\lambda\|S_n - x_n\|_1} = p(\beta) - \phi_\lambda(x). \tag{5.2}$$

The function $I_\beta^{(\lambda)} = p(\beta) - \phi_\lambda(x)$ is nondecreasing in λ. The function

$$I_\beta = \sup_{\lambda>0} I_\beta^{(\lambda)}$$

is then convex and lower semicontinous on B_d with values in $[0, p(\beta)+\ln(2d)]$, and this is the rate function which existence is claimed in Theorem 1.3. We consider now successively $f_n(z) = \mathbf{1}_{\{\|z-nx\|_1\leq n\varepsilon\}}$, where $x\in B_d$ and $\varepsilon > 0$, and $f_n(z) = \mathbf{1}_{\{\|z-nx\|_1< n\varepsilon\}}$.

Lemma 5.2 *For any $x \in B_d$ and any $\varepsilon > 0$, the following limits exist:*

$$L_\beta(x,\varepsilon) = \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \ln \mathbf{E}_n \mathbf{1}_{\{\|S_n - nx\|_1 \le n\varepsilon\}}, \tag{5.3}$$

and

$$L_\beta(x,\varepsilon) = \lim_{n\to+\infty} \frac{1}{n} \ln \mathbf{E}_n \mathbf{1}_{\{\|S_n - nx\|_1 \le n\varepsilon\}} \quad \text{a.s. and in } L^p, \text{ for all } p \ge 1, \tag{5.4}$$

as well as

$$\mathring{L}_\beta(x,\varepsilon) = \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \ln \mathbf{E}_n \mathbf{1}_{\{\|S_n - nx\|_1 < n\varepsilon\}}, \tag{5.5}$$

and

$$\mathring{L}_\beta(x,\varepsilon) = \lim_{n\to+\infty} \frac{1}{n} \ln \mathbf{E}_n \mathbf{1}_{\{\|S_n - nx\|_1 < n\varepsilon\}} \quad \text{a.s. and in } L^p, \text{ for all } p \ge 1. \tag{5.6}$$

Proof of Lemma 5.2. Let us fix $x \in B_d$ and $\varepsilon > 0$. We set

$$Y_n = \mathbf{E}\mathbf{1}_{\{\|S_n - nx\|_1 \le n\varepsilon\}} e^{\beta H_n(S)}, \ v_n = \mathbb{E} \ln Y_n,$$

and show that v_n is superadditive. The method of proof is the same as for proving that $\mathbb{E} \ln Z_n$ is superadditive. If $\|S_n - nx\|_1 \le n\varepsilon$ and $\|(S_{n+m} - S_n) - mx\|_1 \le m\varepsilon$ then $\|S_{n+m} - (n+m)x\|_1 \le (n+m)\varepsilon$, so we have

$$\mathbf{1}_{\{\|S_{n+m} - (n+m)x\|_1 \le (n+m)\varepsilon\}} \ge \mathbf{1}_{\{\|S_n - nx\|_1 \le n\varepsilon\}} \times \mathbf{1}_{\{\|S_{n+m} - S_n - mx\|_1 \le m\varepsilon\}}. \tag{5.7}$$

Therefore

$$Y_{n+m} \ge \mathbf{E}\mathbf{1}_{\{\|S_n - nx\|_1 \le n\varepsilon\}} e^{\beta H_n(S)} \times \mathbf{1}_{\{\|S_{n+m} - S_n - mx\|_1 \le m\varepsilon\}} e^{\beta \sum_{i=n+1}^{n+m} \eta(i,S_i)}. \tag{5.8}$$

The right handside is equal to

$$\sum_{y\in\mathbb{Z}^d} \mathbf{E}\mathbf{1}_{\{\|S_n - nx\|_1 \le n\varepsilon\}} e^{\beta H_n(S)} \mathbf{1}_{\{S_n = y\}} \mathbf{E}^y \mathbf{1}_{\{\|S_m - mx\|_1 \le m\varepsilon\}} e^{\beta \sum_{i=1}^{m} \eta(n+i,S_i)}. \tag{5.9}$$

Let σ_n be the probability measure defined on Σ by

$$\frac{d\sigma_n}{d\mathbf{P}}(S) = \frac{\mathbf{1}_{\{\|S_n - nx\|_1 \le n\varepsilon\}} e^{\beta H_n(S)}}{Y_n}. \tag{5.10}$$

Then (5.8) and (5.9) imply

$$Y_{n+m} \ge \sum_{y\in\mathbb{Z}^d} \sigma_n(S_n = y) Y_n \mathbf{E}^y \mathbf{1}_{\{\|S_m - mx\|_1 \le m\varepsilon\}} e^{\beta \sum_{i=1}^{m} \eta(n+i,S_i)}. \tag{5.11}$$

Using the concavity of the logarithm we obtain

$$\ln Y_{n+m} \ge \ln Y_n + \sum_{y\in\mathbb{Z}^d} \sigma_n(S_n = y) \ln \mathbf{E}^y \mathbf{1}_{\{\|S_m - mx\|_1 \le m\varepsilon\}} e^{\beta \sum_{i=1}^{m} \eta(n+i,S_i)}. \tag{5.12}$$

We take the conditional expectation with respect to $\mathbb{P}$ given $\mathcal{F}_n$ (recall that $\mathbb{E}_n$ denotes the conditional expectation with respect to $\mathbb{P}$ given $\mathcal{F}_n$, $\mathcal{F}_n = \sigma(\eta(i,x) : 1 \le i \le n, x \in \mathbb{Z}^d)$, and we obtain the following inequality

$$\mathbb{E}_n \ln Y_{n+m} \ge \ln Y_n + \mathbb{E} \ln Y_m. \tag{5.13}$$

Integrating with respect to $\mathbb{P}$ we conclude that (v_n) is superadditive:

$$v_{n+m} \geq v_n + v_m.$$

Therefore the limit

$$l_\beta(x,\varepsilon) = \lim_{n\to+\infty} \frac{v_n}{n} = \sup_{n\geq 1} \frac{v_n}{n} \tag{5.14}$$

exists, and so does, by Corollary 4.2, the limit

$$L_\beta(x,\varepsilon) = \lim_{n\to+\infty} \frac{1}{n} \ln \mathbf{E}_n \mathbf{1}_{\{\|S_n - nx\|_1 \leq n\varepsilon\}} \ \mathbb{P}\ a.s. \text{ and in } L^p \text{ for all } p \geq 1, \tag{5.15}$$

with

$$L_\beta(x,\varepsilon) = l_\beta(x,\varepsilon) - p(\beta).$$

The proof of (5.5) and (5.6) is the same as for (5.3) and (5.4). ■

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